Final Exam Solutions

Introduction to Statistics(통계의 이해 영어강의) **2026 1st semester**

Section(교반): _____ Cadet Number(교번): __________ Name(성명): ______________ Score: ____________

- All solutions must include a detailed step-by-step explanation.
- If an answer has more than four decimal places, round to the **fourth decimal place**.
- **Reference materials** are provided on the last page of the exam.

1. Read each statement below. Using the word box in the Reference Materials, fill in each blank with the appropriate term, or determine whether the statement is **True (O)** or **False (X)**. [8 points]

(1) A numerical value describing a characteristic of the population is called a (____________________).

(2) Suppose a random sample $X_1, \ldots, X_n$ of size n is taken from a Bernoulli distribution $B(1, p)$ with probability of success p. A point estimate of the population proportion p is (____________________), $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i$

(3) The rejection region is the set of test statistic values for which H_0 is rejected with a certain probability α. This probability α is called the (____________________).

(4) The (____________________) of the i-th observation (x_i, y_i) is the difference between the observed y_i and the predicted $\hat{y}_i$.

(5) The central limit theorem states that, regardless of the population distribution, the distribution of the sample mean $\bar{X}$ approaches a normal distribution as the sample size n increases. (O / X)

(6) If $X_1, \ldots, X_n$ is a random sample from a normal population $N(\mu, \sigma^2)$ and S^2 is the sample variance, then $V = \dfrac{(n-1)S^2}{\sigma^2}$ follows a chi-squared distribution with n degrees of freedom. (O / X)

(7) For two estimators $\hat{\theta}_1, \hat{\theta}_2$ of a parameter θ, if $\mathrm{Var}(\hat{\theta}_1) < \mathrm{Var}(\hat{\theta}_2)$, then $\hat{\theta}_1$ is said to be more efficient than $\hat{\theta}_2$. (O / X)

(8) Least squares regression tries to maximize the sum of squared errors. (O / X)

<u>Solution:</u> parameter / sample proportion / significance level / residual / O / X / O / X

2. A random sample $X_1, X_2, \ldots, X_n$ of size n is drawn from a population with probability density function:

$$f(x) = \begin{cases} \dfrac{1}{2\theta}, & 0 < x < 2\theta, \\ 0, & \text{otherwise,} \end{cases}$$

with $\theta > 0$. Answer the following. [10 points]

(You may use $E(X_1) = \theta$ and $Var(X_1) = \dfrac{\theta^2}{3}$)

(1) Show that the sample mean $\bar{X}$ is an unbiased estimator of θ.

<u>Solution:</u>

$$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n} \cdot n \cdot \theta = \theta.$$

Therefore the sample mean $\bar{X}$ is an unbiased estimator of θ.

(2) Show that the sample mean $\bar{X}$ is a consistent estimator of θ.

<u>Solution:</u> First, $\lim\limits_{n\to\infty} E(\bar{X}) = \lim\limits_{n\to\infty} \theta = \theta$.

$$\mathrm{Var}(\bar{X}) = \frac{1}{n}\mathrm{Var}(X_i) = \frac{\theta^2}{3n}.$$

$$\lim_{n\to\infty} \mathrm{Var}(\bar{X}) = \lim_{n\to\infty} \frac{\theta^2}{3n} = 0.$$

Since $\lim_{n\to\infty} E(\bar{X}) = \theta$ and $\lim_{n\to\infty} \mathrm{Var}(\bar{X}) = 0$, the sample mean $\bar{X}$ is a consistent estimator of θ.

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3. A buyer wants to test whether the average diameter of nuts from a supplier is different from $\mu_0 = 12.7$ mm. A random sample of $n = 10$ nuts is selected and their diameters $X_1, X_2, \cdots, X_{10}$ are measured. Test at significance level $\alpha = 0.05$. Assume the nut diameters follow a normal distribution. [12 points]

(1) State the null and alternative hypotheses.

Solution:

$$H_0 : \mu = 12.7, \qquad H_A : \mu \neq 12.7.$$

(2) Find the test statistic and its null distribution.

Solution: Since the population variance σ^2 is unknown, we use the T statistic. When H_0 is true,

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{\bar{X} - 12.7}{S/\sqrt{10}} \sim t(9).$$

(3) The measured diameters of the 10 selected nuts are shown below. Find the observed value of the test statistic.

Nut (i)	1	2	3	$\cdots$	8	9	10
Diameter $(x_i,\ \mathrm{mm})$	12.68	12.79	12.85	$\cdots$	12.67	12.79	12.65

※ $\bar{x} = \frac{1}{10}\sum_{i=1}^{10} x_i = 12.744, \quad s = \sqrt{\frac{1}{10-1}\sum_{i=1}^{10}(x_i - \bar{x})^2} = 0.0783$

Solution:

$$\bar{x} = \frac{1}{10}\sum_{i=1}^{10} x_i = \frac{127.44}{10} = 12.744, \qquad s = \sqrt{\frac{1}{10-1}\sum_{i=1}^{10}(x_i - \bar{x})^2} = \sqrt{\frac{0.0552}{9}} = 0.0783.$$

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{12.744 - 12.7}{0.0783/\sqrt{10}} = 1.7770.$$

(4) Using the result in (3), report the p-value and complete the hypothesis test. State the conclusion in the context of the data.

Solution: Since $H_A : \mu \neq 12.7$ is a two-sided test,

$$p\text{-value} = 2\,P(T > |t|) = 2\,P(T > 1.7770) = 2 \times 0.0546 = 0.1092.$$

Since $p\text{-value} = 0.1092 > \alpha = 0.05$, we fail to reject H_0.

Conclusion: At significance level $\alpha = 0.05$, there is insufficient evidence to conclude that the average diameter of the delivered nuts differs from the required specification of 12.7 mm.

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4. The average recovery time for an existing health supplement is $\mu_0 = 20$ days. A team claims that their new supplement gives a smaller average recovery time (less than 20 days). To test this, $n = 16$ people take the new supplement and their recovery times $X_1, X_2, \cdots, X_{16}$ are measured. Test at significance level $\alpha = 0.05$. Assume the recovery time follows a normal distribution with standard deviation $\sigma = 4$ days. [12 points]

(1) State the null and alternative hypotheses.

Solution:

$$H_0 : \mu = 20, \qquad H_A : \mu < 20.$$

(2) Find the test statistic and its null distribution.

Solution: In a normal population with known variance σ^2, the test statistic for the population mean and its distribution are

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{X} - 20}{4/\sqrt{16}} \sim N(0,\, 1).$$

(3) Find the rejection region at significance level $\alpha = 0.05$.

Solution: Since this is a left-tailed test, the rejection region is

$$z < -z_{0.05} = -1.6449.$$

(4) The sample mean recovery time is $\bar{x} = 18.2$ days. Find the observed value of the test statistic.

Solution:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{18.2 - 20}{4/\sqrt{16}} = -1.8.$$

(5) Using the result in (4) and the rejection region, complete the hypothesis test. State the conclusion in the context of the data.

Solution: The observed value $z = -1.8$ lies in the rejection region $z < -1.6449$, so we reject H_0.

Conclusion: At significance level $\alpha = 0.05$, there is sufficient evidence to support the claim that the new health supplement gives a smaller average recovery time (less than 20 days).

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5. A company claims that its new earplugs reduce noise by more than $\delta_0 = 2$ dB on average compared with the existing earplugs. To test this, $n_A = 40$ new and $n_B = 50$ existing earplugs are randomly selected, and their noise reductions $X_{A1}, \ldots, X_{A40}$ and $X_{B1}, \ldots, X_{B50}$ are measured. Test at significance level $\alpha = 0.01$. Assume the noise reductions of the new and existing earplugs follow normal distributions with means μ_A, μ_B and standard deviations $\sigma_A = 0.5, \sigma_B = 0.6$. [15 points]

(1) State the null and alternative hypotheses.

Solution:

$$H_0 : \mu_A - \mu_B = 2, \qquad H_A : \mu_A - \mu_B > 2.$$

(2) $\bar{X}_A - \bar{X}_B$ is used as the test statistic for (1), where $\bar{X}_A$ and $\bar{X}_B$ are the sample means for the new and existing earplugs, respectively. Find the null distribution of this test statistic.

Solution: For two normal populations with known variances σ_A^2 and σ_B^2, the variance of the test statistic $\bar{X}_A - \bar{X}_B$ is $\dfrac{\sigma_A^2}{n_A} + \dfrac{\sigma_B^2}{n_B}$. Therefore its distribution is

$$\bar{X}_A - \bar{X}_B \sim N\left(2,\ \frac{0.25}{40} + \frac{0.36}{50}\right).$$

(3) Find the rejection region at significance level $\alpha = 0.01$.

Solution: This is a right-tailed test and $z_{0.01} = 2.3263$, so the rejection region is

$$\bar{x}_A - \bar{x}_B > 2 + 2.3263 \times \sqrt{\frac{0.25}{40} + \frac{0.36}{50}} = 2.2698.$$

(4) The sample mean noise reductions of the selected new and existing earplugs are $\bar{x}_A = 27.48$ and $\bar{x}_B = 25.17$. Find the observed value of the test statistic and test the hypothesis at significance level $\alpha = 0.01$. State the conclusion in the context of the data.

Solution: The observed value of the test statistic is $\bar{x}_A - \bar{x}_B = 27.48 - 25.17 = 2.31$.

Since 2.31 lies in the rejection region $\bar{x}_A - \bar{x}_B > 2.2698$, we reject H_0.

Conclusion: At significance level $\alpha = 0.01$, there is sufficient evidence that the new earplugs reduce noise by more than 2 dB on average compared with the existing earplugs.

(5) When the true difference of the two population means is $\mu_A - \mu_B = 2.5$, find the power of the test.

Solution: Since the rejection region is $\bar{x}_A - \bar{x}_B > 2.2698$, the power is

$$\begin{aligned}
1 - \beta &= P\left(\bar{X}_A - \bar{X}_B > 2.2698 \,\middle|\, \mu_A - \mu_B = 2.5\right) \\
&= P\left(\frac{\bar{X}_A - \bar{X}_B - 2.5}{\sqrt{\frac{0.25}{40} + \frac{0.36}{50}}} > \frac{2.2698 - 2.5}{\sqrt{\frac{0.25}{40} + \frac{0.36}{50}}}\right) \\
&= P(Z > -1.9850) = P(Z \le 1.9850) = 0.9764.
\end{aligned}$$

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6. A newly developed protein supplement is tested to see whether it increases grip strength (kg). For $n = 12$ randomly selected people, grip strength is measured before, X_{Ai}, and after, X_{Bi} $(i = 1, \ldots, 12)$, taking the supplement. Using a paired test at significance level $\alpha = 0.05$, we test whether the average grip strength is greater after than before. Assume the before, X_A, and after, X_B, grip strengths each follow a normal distribution with means μ_A and μ_B, and define $\mu_{\text{diff}} = \mu_A - \mu_B$. The hypotheses are $H_0 : \mu_{\text{diff}} = 0$ and $H_A : \mu_{\text{diff}} < 0$. [14 points]

(1) Find the test statistic and its null distribution.

Solution: Let $X_{\text{diff}} = X_A - X_B$ be the paired difference. Under H_0, the test statistic and its null distribution are

$$T = \frac{\bar{X}_{\text{diff}} - 0}{S_{\text{diff}}/\sqrt{n_{\text{diff}}}} = \frac{\bar{X}_{\text{diff}}}{S_{\text{diff}}/\sqrt{12}} \overset{H_0}{\sim} t(11),$$

where $\bar{X}_{\text{diff}}$ and S_{diff} are the sample mean and sample standard deviation of the $n_{\text{diff}} = 12$ paired differences.

(2) Find the rejection region at significance level $\alpha = 0.05$.

Solution:

$$t < -t_{0.05}(11) = -1.7959.$$

(3) The partial R output for this problem is shown below. Using the partial R output and the rejection region in (2), complete the hypothesis test. State the conclusion in the context of the data.

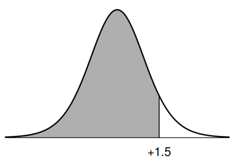
```
       Paired t-test

data:  before and after
t = -1.5, df = __________ , p-value = ______________
alternative hypothesis: true mean difference is less than 0
95 percent confidence interval:
 -Inf 0.5917717
sample estimates:
mean difference
             -3
```

Solution: The observed value of the test statistic is $t = -1.5$, which does not lie in the rejection region $t < -1.7959$, so we fail to reject H_0.

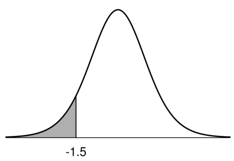
Conclusion: At significance level $\alpha = 0.05$, there is insufficient evidence to support the claim that the grip strength after taking the supplement is greater than before.

(4) Among the figures below, choose the one whose shaded (gray) region best corresponds to the p-value in (3).

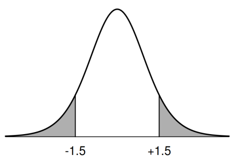
①



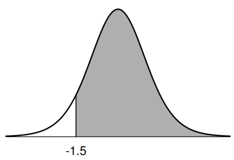
②



③



④



Solution: ② (The p-value is the left-tail probability $P(T < -1.5)$.)

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7. A café headquarters wants to test whether there is a correlation (ρ) between the floor area (X) and the weekly sales (Y) of a branch. A random sample of $n = 10$ branches gives the results below. Test at significance level $\alpha = 0.05$. Assume (X, Y) follows a bivariate normal distribution. [14 points]

Floor area $(x_i,\ \text{pyeong})$	14	19	25	$\cdots$	26	17	20
Weekly sales $(y_i,\ 10{,}000\ \text{KRW})$	245	275	378	$\cdots$	390	268	292

(1) State the null and alternative hypotheses.

Solution:

$$H_0 : \rho = 0, \qquad H_A : \rho \neq 0.$$

(2) The test statistic for (1) is $\dfrac{R\sqrt{n-2}}{\sqrt{1-R^2}}$, where R is the sample correlation coefficient. Find the null distribution of this test statistic.

Solution: With $n = 10$, when H_0 is true,

$$T = \frac{R\sqrt{n-2}}{\sqrt{1-R^2}} = \frac{R\sqrt{8}}{\sqrt{1-R^2}} \sim t(8).$$

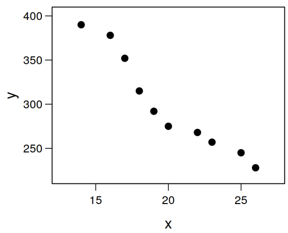
(3) Find the rejection region at significance level $\alpha = 0.05$.

Solution: This is a two-sided test, so the rejection region is

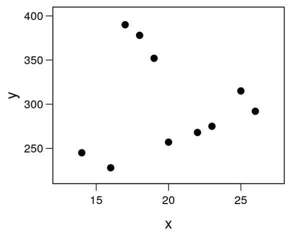
$$|t| > t_{0.025}(8) = 2.3060.$$

(4) When the observed sample correlation coefficient is $r = 0.9652$, choose the figure below that best represents the scatter plot of the floor area (X) and the weekly sales (Y).

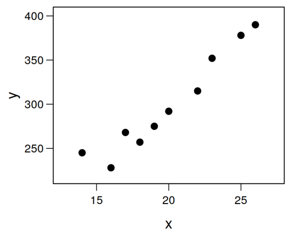
①



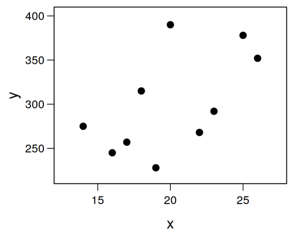
②



③



④



Solution: ③ (Since $r = 0.9652$ is close to 1, the points show a strong positive linear pattern.)

(5) When the observed sample correlation coefficient is $r = 0.9652$, find the observed value of the test statistic and test the hypothesis using the rejection region in (3). State the conclusion in the context of the data.

Solution: Since $r = 0.9652$, the observed value of the test statistic is

$$t = \frac{r\sqrt{8}}{\sqrt{1-r^2}} = 10.4393.$$

Since this lies in the rejection region $|t| > t_{0.025}(8) = 2.3060$, we reject H_0.

Conclusion: At significance level $\alpha = 0.05$, there is sufficient evidence of a correlation between the floor area and the weekly sales.

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8. A shooting instructor analyzes the relationship between a recruit's practice time (x, hours) and shooting-test score (Y, points) using the simple linear regression model $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where $\epsilon_i \sim N(0, \sigma^2)$. A random sample of $n = 10$ recruits is analyzed using the R function `lm()`, with the results shown below. Answer the following. [15 points]

```
> x <- c(10, 12, 14, ..., 24, 26, 28)
> y <- c(64, 80, 60, ..., 94, 80, 97)
> summary(lm(y ~ x))
```

```
Call:
lm(formula = y ~ x)

Residuals:
    Min      1Q  Median      3Q     Max
-12.030  -8.614  -1.500   8.462  10.758

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  52.5152    11.0408   4.756  0.00143 **
x             1.3939     0.5562   2.506  0.03659 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 10.1 on 8 degrees of freedom
Multiple R-squared:  0.4398,    Adjusted R-squared:  0.3698
F-statistic:  6.28 on 1 and 8 DF,  p-value: 0.03659
```

(1) Using the R output, find the values in the table below.

Estimated regression coefficients		**Estimated regression equation**
$\hat{\beta}_0$	$\hat{\beta}_1$	

<u>Solution:</u>

$\hat{\beta}_0 = 52.5152$, $\hat{\beta}_1 = 1.3939$, and the estimated regression equation is $\hat{Y} = 52.5152 + 1.3939\,x$.

(2) The first recruit's practice time is $x_1 = 10$ and shooting-test score is $y_1 = 64$. Using the simple linear regression analysis, find the fitted value $\hat{y}_1$ and the residual e_1.

<u>Solution:</u> By the estimated regression equation, the fitted value is

$$\hat{y}_1 = \hat{\beta}_0 + \hat{\beta}_1 x_1 = 52.5152 + 1.3939 \times 10 \approx 66.4542,$$

and the residual is

$$e_1 = y_1 - \hat{y}_1 = 64 - 66.4542 = -2.4542.$$

(3) Suppose $\hat{\sigma}^2 = MSE = 102.01$. Find $\sum_{i=1}^{10} e_i^2$ for the residuals $e_1, \ldots, e_{10}$.

<u>Solution:</u>

$$\sum_{i=1}^{10} e_i^2 = SSE = (10-2) \times MSE = 8 \times 102.01 = 816.08.$$

(4) Using the R output, we test the hypothesis $H_0 : \beta_1 = 0$ vs. $H_A : \beta_1 \neq 0$ at significance level $\alpha = 0.05$ to see whether the practice time affects the shooting-test score.

a. The observed value of the test statistic is (________), and its p-value is (________). Therefore, at significance level $\alpha = 0.05$, we (________) H_0.

b. State the conclusion in the context of the data.

<u>Solution:</u>

a. test statistic $= 2.506$, $p\text{-value} = 0.03659$; therefore we **reject** H_0 at $\alpha = 0.05$.

b. At significance level $\alpha = 0.05$, the explanatory variable (practice time) is a statistically significant predictor of the response variable (shooting-test score).