Chapter 4: Random variables

Yonghyun Kwon
Department of Mathematics, Korea Military Academy

Defining random variables

Random variables

- A *random variable(RV)* is a numeric quantity whose value depends on the outcome of a random event
  - We use a capital letter, like X, to denote a random variable
  - The values of a random variable are denoted with a lowercase letter, in this case x
  - For example, $P(X = x)$
- There are two types of random variables:
  - *Discrete random variables* often take only integer values

    Countable

    - Example: Number of credit hours, Difference in number of credit hours this term vs last
  - *Continuous random variables* take real (decimal) values

    uncountable

    - Example: Cost of books this term, Difference in cost of books this term vs last

Let a random variable X denote the number of heads in two coin tosses. Find the probability distribution of X.

	HH	TH	HT	TT
X	2	1	1	0

$P(X=2) = \frac{1}{4}$

$P(X=1) = \frac{2}{4}$

$P(X=0) = \frac{1}{4}$

$f(x) = P(X=x)$

- Probability distribution function:

$$f(x) = \begin{cases} 1/4 & x=0 \\ 1/2 & x=1 \\ 1/4 & x=2 \\ 0 & \text{o.w.} \end{cases}$$

- Probability distribution table:

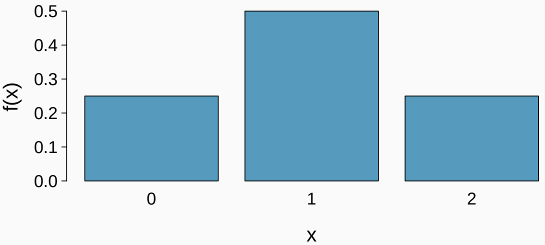
x	0	1	2	o.w.
$f(x)$	1/4	1/2	1/4	0

Probability mass function(pmf), $f(x)$ (e.g. $x_1 = 0$, $x_2 = 1$, $x_3 = 2$)

$$f(x) = \begin{cases} P(X = x_i), & x = x_i \; (i = 1, \cdots, k) \\ 0, & o.w. \end{cases}$$

Properties of probability mass function

- $0 \leq f(x) \leq 1$, $\quad \sum_{i=1}^{k} f(x_i) = 1$.
- $P(a < X < b) = \sum_{a<x<b} f(x)$.



Continuous random variable

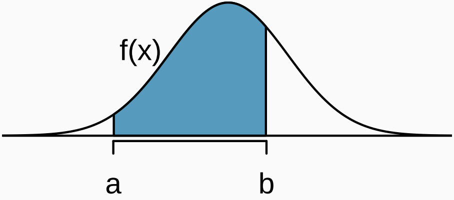
Probability density function(pdf), $f(x)$

$$P(a \le X \le b) = \int_a^b f(x)dx$$

$$1 = P(-\infty \le X \le \infty) = \int_{-\infty}^{\infty} f(x)\,dx$$

Properties of probability density function

- $f(x) \ge 0$, $\int_{-\infty}^{\infty} f(x)dx = 1$.
- $P(a < X < b) = P(a \le X \le b) = \int_a^b f(x)dx$.



Practice

Consider a continuous RV X with a probability density function $f(x) = \begin{cases} cx(1-x), & 0 \le x \le 1 \\ 0, & o.w. \end{cases}$, where c is a constant.

- Find a constant c so that $f(x)$ is a probability density function.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{0} + \int_{0}^{1} f(x)\,dx + \underbrace{\int_{1}^{\infty} f(x)\,dx}_{=0} = \int_{0}^{1} cx(1-x)\,dx$$

$$= c\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{c}{6} \Rightarrow c = 6$$

- Find $P(X \le 1/2)$.

$$= P(-\infty \le X \le 1/2)$$

$$= \int_{-\infty}^{1/2} f(x)\,dx = \int_{0}^{1/2} 6x(1-x)\,dx = \frac{1}{2}$$

Expectation and variance

If you repeat tossing two coins, how many heads would you expect to appear on average?

Expectation

We are often interested in *average* outcome of a random variable.

Expectation(mean) of a RV X

$$\mu = E(X) = \begin{cases} \sum_{i=1}^{k} x_i f(x_i) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

Expectation of a function of RV $g(X)$

$$E\left(g(X)\right) = \begin{cases} \sum_{i=1}^{k} g(x_i) f(x_i) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

Let a random variable X denote the number of heads in two coin tosses. Find the expectation of X.

The probability distribution of X is given as

x	0	1	2
$f(x)$	1/4	2/4	1/4

- $E(X) = \sum_{\forall x} x f(x) = 0 \cdot f(0) + 1 \cdot f(1) + 2 \cdot f(2) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 2 \cdot \frac{1}{4} = 1$

Find the expectation of $Y = (X-1)^2$.

- $E(Y) = E\left[(X-1)^2\right] =$

$$= \sum_{\forall x} (x-1)^2 f(x) = (0-1)^2 f(0) + (1-1)^2 f(1) + (2-1)^2 f(2)$$

$$= 1 \times \frac{1}{4} + 1 \times \frac{1}{4} = \frac{1}{2}$$

Variance and standard deviation

We are also often interested in the *variability* of a random variable.

Variance and standard deviation of a RV X

$$\sigma^2 = Var(X) = E[(X-\mu)^2] = \begin{cases} \sum_{i=1}^{k} (x_i - \mu)^2 f(x_i) & X \text{ discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx & X \text{ continuous} \end{cases}$$

$$\sigma = SD(X) = \sqrt{Var(X)}$$

Computing variance

$$Var(X) = E(X^2) - E(X)^2$$

Note:

$$Var(X) = E[(X-\mu)^2] = E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - E(X)^2$$

Practice

Tony is waiting for a bus. Suppose that waiting time X (hours) has the following a probability density function $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & o.w. \end{cases}$

- Find $E(X)$: $= \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

- Find $Var(X)$:

$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

Properties of expectation and variance

Let a and b be constants(not random).

Property of expectation

$E(aX + b) = aE(X) + b$

Properties of variance

- $Var(aX + b) = a^2 Var(X)$
- $Var(X) \geq 0$

$\underbrace{Var(X)}_{\geq 0} = E\underbrace{(X-\mu)^2}_{\geq 0}$

Note: *If X is a continuous random variable(discrete X can be shown similarly),*

$$E(aX + b) = \int_{-\infty}^{\infty} (ax + b)f(x)dx = a\int_{-\infty}^{\infty} xf(x)dx + b\int_{-\infty}^{\infty} f(x)dx = aE(X) + b$$

$$Var(aX + b) = E\left(aX + b - E(aX + b)\right)^2 = E\left(aX + b - aE(X) - b\right)^2$$

$$= a^2 E\left(X - E(X)\right)^2 = a^2 Var(X)$$

Practice

Tony is waiting for a bus. Suppose that waiting time X (hours) has the following a probability density function $f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & o.w. \end{cases}$

Let $Y = 2X + 1$.

$E(X) = \frac{1}{2}$ $Var(X) = \frac{1}{12}$

- Find $E(Y)$: $= E(2X+1) = 2E(X) + 1 = 2 \times \frac{1}{2} + 1 = 2$

- Find $Var(Y)$:

  $= Var(2X+1) = 4Var(X) = 4 \times \frac{1}{12} = \frac{1}{3}$

Joint distributions

Recall: Conditional probability

The conditional probability of the outcome of interest A given condition B is calculated as

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \quad \begin{array}{l} \leftarrow \text{joint probability} \\ \leftarrow \text{marginal probability} \end{array}$$

Joint probability: Probability for two events(variables).

Marginal probability: Probability for a single event(variable).

Joint probability distribution

$1 = P(-\infty \le X \le \infty, -\infty \le Y \le \infty)$
$= \sum_{\forall x} \sum_{\forall y} f(x,y)$

Joint pmf of two *discrete* RVs X and Y

$$f(x,y) = P(X = x, Y = y)$$

- $0 \le f(x,y) \le 1$, $\sum_{\forall x} \sum_{\forall y} f(x,y) = 1$.
- $P(a < X \le b, c < Y \le d) = \sum_{a<x\le b} \sum_{c<y\le d} f(x,y)$

Joint pdf of two *continuous* RVs X and Y

$$\int_a^b \int_c^d f(x,y) dy dx = P(a < X < b, c < Y < d)$$

- $0 \le f(x,y)$, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$.

$\iint_{(x,y)\in A} f(x,y)\, dx dy = P((X,Y) \in A)$

$f_X(x) = P(X = x, -\infty \le Y \le \infty)$

$= \sum_{\forall y} f(x,y) \left(= \sum_{-\infty \le y \le \infty} f(x,y) \right)$.

Marginal pmf of a *discrete* RV

$$f_X(x) = \sum_{\forall y} f(x,y), \quad f_Y(y) = \sum_{\forall x} f(x,y)$$

Marginal pdf of a *continuous* RV

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

Practice

In a certain suburban area, each household reported the number of cars X and the number of television sets Y that they owned. Joint pmf of X and Y is given below.

$$f_X(x) = \sum_{\forall y} f(x,y)$$

X \ Y	1	2	3	4
1	0.1	0.0	0.1	0.0
2	0.3	0.0	0.1	0.2
3	0.0	0.2	0.0	0.0

- Find the marginal pmf of X:

x	1	2	3
$f_X(x)$	0.2	0.6	0.2

- Find the marginal pmf of Y:

y	1	2	3	4
$f_Y(y)$	0.4	0.2	0.2	0.2

Recall: Independence of two events

Two events A and B are independent if

$$P(A \text{ and } B) = P(A) \times P(B).$$

Independence of two random variables

X and Y are (mutually) independent if

$$f(x, y) = f_X(x) f_Y(y).$$

Practice

Joint pdf of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{3x-y}{9} & 1 < x < 3,\ 1 < y < 2 \\ 0, & o.w. \end{cases}$$

Determine whether X and Y are independent.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_1^2 \frac{3x-y}{9}\,dy = \int_1^2 \frac{3x}{9}\,dy - \int_1^2 \frac{y}{9}\,dy$$

$$= \frac{3x}{9}\int_1^2 1\,dy - \frac{1}{9}\int_1^2 y\,dy = \frac{3x}{9} - \frac{1}{9} \times \frac{3}{2} = \frac{1}{3}x - \frac{1}{6} \quad (1 < x < 3)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{4}{3} - \frac{2}{9}y \quad (1 < y < 2)$$

$f(x,y) \neq f_X(x) f_Y(y) \Rightarrow X$ and Y are not independent

Covariance

Expectation of a function of RVs $g(X, Y)$

$$E\left(g(X,Y)\right) = \begin{cases} \sum_{\forall x} \sum_{\forall y} g(x,y) f(x,y) & \text{if } X, Y \text{ discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y) dxdy & \text{if } X, Y \text{ continuous} \end{cases}$$

Covariance measures the joint variability of two random variables.

Covariance

Covariance of two random variables X and Y is

$$\begin{aligned} Cov(X,Y) &= E\left[(X - E(X))(Y - E(Y))\right] \\ &= E(XY) - E(X)E(Y). \end{aligned}$$

Note: If X and Y are independent, $Cov(X,Y) = 0$, but not vice versa.

Practice

$E(X) = \sum_{\forall x} x f(x)$

Joint and marginal pmf of X and Y are given below.

X \ Y	1	2	3	4	$f(x)$
1	0.1	0.0	0.1	0.0	0.2
2	0.3	0.0	0.1	0.2	0.6
3	0.0	0.2	0.0	0.0	0.2
$f(y)$	0.4	0.2	0.2	0.2	1.0

- Find $E(X)$ and $E(Y)$:

$E(X) = 1 \times 0.2 + 2 \times 0.6 + 3 \times 0.2 = 2$

$E(Y) = 1 \times 0.4 + 2 \times 0.2 + 3 \times 0.2 + 4 \times 0.2$
$= 2.2$

- Find $E(XY)$:

$E(XY) = \sum_{\forall x} \sum_{\forall y} xy\, f(x,y)$

$= 1 \times 1 \times 0.1 + 1 \times 3 \times 0.1$

$+ 2 \times 1 \times 0.3 + 2 \times 3 \times 0.1 + 2 \times 4 \times 0.2$

$+ 3 \times 2 \times 0.2 \quad = 4.4$

- Find $Cov(X, Y)$:

$= E(XY) - E(X)E(Y) = 4.4 - 2 \times 2.2$
$= 0$

Note: Are X and Y independent?

$f(x,y) = f_X(x)\, f_Y(y)$?

$f(1,1) \neq f_X(1) \cdot f_Y(1) \Rightarrow X, Y$ are not independent

Correlation

Correlation of two random variables X and Y is

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}.$$

Properties of correlation

- $-1 \leq \rho(X, Y) \leq 1$
- $\rho(X, Y) > 0$: positive linear relationship between X and Y.
- $\rho(X, Y) < 0$: negative linear relationship between X and Y.
- $\rho(X, Y) = 0$: no linear relationship between X and Y.

Linear combinations

A *linear combination* of random variables X and Y is given by

$$aX + bY$$

$E(aX+b) = aE(X)+b$

where a and b are some fixed constants. $Var(aX+b) = a^2 Var(X)$

The average value of a linear combination of RVs is given by

Expectation of a linear combination of two RVs

$$E(aX + bY) = a \times E(X) + b \times E(Y)$$

The variance of a linear combination of two *independent* RVs is

Variance of a linear combination of two RVs

$$Var(aX + bY) = a^2 \times Var(X) + b^2 \times Var(Y)$$

Practice

On average a notebook costs \$10 and a pen costs \$15. You plan to buy 5 notebooks (all at the same random price X) and 4 pens (all at the same random price Y). What is the expected total cost $E(5X + 4Y)$?

$E(5X+4Y) = 5E(X) + 4E(Y) = 5\times 10 + 4\times 15 = 110$

The standard deviation of the price is \$1.5 for a notebook and \$2 for a pen. Assuming independence between X and Y, what is the standard deviation of the total cost $SD(5X + 4Y)$? $SD(X) = \sqrt{Var(X)}$

$Var(5X+4Y) = 25\,Var(X) + 16\,Var(Y)$

$= 25\times 1.5^2 + 16\times 2^2 = 120.25$

$\Rightarrow SD(5X+4Y) = \sqrt{120.25} = 10.97$

Simplifying random variables

$2X = X + X$ can be understood as a linear combination of two random variables X and X. Thus,

$$E(2X) = E(X + X) = E(X) + E(X) = 2E(X),$$

but

$$Var(X + X) \neq Var(X) + Var(X),$$

because X and X themselves are not independent. Instead,

$$Var(X + X) = Var(2X) = \underline{4\,Var(X)}$$

Variance of Linear Combinations: General Case

The variance formula for $aX + bY$ assumed independence. In general, covariance must be included.

Variance of a linear combination (general)

$$Var(X \pm Y) = Var(X) + Var(Y) \pm 2\,Cov(X, Y)$$

$$Var(aX \pm bY) = a^2\,Var(X) + b^2\,Var(Y) \pm 2ab\,Cov(X, Y)$$

Note: When X and Y are independent, $Cov(X, Y) = 0$, and the formula reduces to

$$Var(aX + bY) = a^2\,Var(X) + b^2\,Var(Y),$$

which is the formula from the previous slide.

Exercises

Exercises in OpenIntro Statistics 4th ed.

- **Conditional probability:**
  - Exercise 3.22, 3.42.
- **Random variables:**
  - Exercise 3.34, 3.44 (a) and (b), 3.47(or 3.45)

1. Suppose that X and Y have a joint probability density function

$$f(x,y) = \begin{cases} \frac{3}{2}y^2, & 0 \le x \le 2, 0 \le y \le 1 \\ 0, & o.w. \end{cases}$$

1. Find the marginal pdf of Y, $f_Y(y)$.
2. Find $E(Y)$ and $Var(Y)$.

Exercises(cont.)

2. Suppose that X and Y have a joint probability mass function

$$f(x, y) = \begin{cases} \frac{1}{12}(x + y), & x = 0, 2, \text{ and } y = 0, 1, 2, \\ 0, & o.w. \end{cases}$$

1. Find the marginal pmf of X, $f_X(x)$. (Hint: Use a table in slide 15).
2. Find the marginal pmf of Y, $f_Y(y)$.
3. Find $E(X)$ and $E(Y)$.
4. Find $E(XY)$ and $Cov(X, Y)$.

3. Suppose that X and Y have a joint probability density function

$$f(x,y) = \begin{cases} e^{-x-y}, & x > 0,\ y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.
2. Find $E(X)$ and $E(Y)$.
3. Find $E(XY)$ and $\mathrm{Cov}(X,Y)$.
4. Find $P(X < Y)$.