Introduction to Statistics - Homework #9

- Use Appendix C: Distribution tables (e.g. Z-table) from page 408 to 417 in the textbook if necessary.

Practice

A nutrition researcher wants to test whether a new diet increases average daily protein intake. Group 1 follows the new diet, while Group 2 follows their usual eating habits. To test this at the significance level $\alpha = 0.05$, the researcher samples $n_1 = 16$ diet participants and $n_2 = 12$ control participants. Assume the intakes follow:

$$X_{11}, \cdots, X_{1,16} \sim N(\mu_1, 64), \qquad X_{21}, \cdots, X_{2,12} \sim N(\mu_2, 144),$$

where μ_1 and μ_2 denote the mean protein intakes for Group 1 and Group 2, respectively.

(1) State the null and alternative hypotheses.

Solution: The diet is expected to *increase* average protein intake. Let μ_1 and μ_2 be the group means.

$$H_0 : \mu_1 - \mu_2 = 0, \qquad H_A : \mu_1 - \mu_2 > 0$$

(2) Find the test statistic and its null distribution.

Solution 1: The sampling distributions are

$$\bar{X}_1 \sim N\left(\mu_1, \frac{64}{16}\right) = N(\mu_1, 4), \qquad \bar{X}_2 \sim N\left(\mu_2, \frac{144}{12}\right) = N(\mu_2, 12).$$

Thus, the test statistic and its null distribution are:

$$\bar{X}_1 - \bar{X}_2 \sim N(0, 4^2).$$

Solution 2: Using the same sampling distributions as above, we standardize the difference to obtain the Z-statistic and its null distribution:

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - 0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{64}{16} + \frac{144}{12}}} = \frac{\bar{X}_1 - \bar{X}_2}{4} \sim N(0, 1).$$

(3) Find the rejection region.

Solution 1: Note that $z = \frac{\bar{x}_1 - \bar{x}_2}{4}$. We reject H_0 if

$$\frac{\bar{x}_1 - \bar{x}_2}{4} > 1.645 \iff \bar{x}_1 - \bar{x}_2 > 1.645 \cdot 4 = 6.58.$$

Thus, the rejection region is $\bar{x}_1 - \bar{x}_2 > 6.58$.

Solution 2: Using the Z-statistic from Solution 2 of (2), reject H_0 if

$$z > z_\alpha = 1.645.$$

Thus, the rejection region is $z > 1.645$.

(4) Given $\bar{x}_1 = 80$ and $\bar{x}_2 = 74$, complete the hypothesis test.

Solution 1:

$$\bar{x}_1 - \bar{x}_2 = 80 - 74 = 6.$$

Since $6 < 6.58$, the observed difference does *not* fall in the rejection region. Therefore, we do not reject H_0.

Conclusion: At the significance level $\alpha = 0.05$, there is not sufficient evidence to conclude that the new diet increases average daily protein intake.

Solution 2:

$$z = \frac{\bar{x}_1 - \bar{x}_2}{4} = \frac{80 - 74}{4} = \frac{6}{4} = 1.5.$$

Since $1.5 < 1.645$, the observed Z-value does *not* fall in the rejection region. Therefore, we do not reject H_0.

Conclusion: At the significance level $\alpha = 0.05$, there is not sufficient evidence to conclude that the new diet increases average daily protein intake.

(5) Find the power of the test assuming $\mu_1 - \mu_2 = 10$.

Solution:

$$\begin{aligned}
P(\text{Reject } H_0 \mid H_A \text{ is true}) &= P(\bar{X}_1 - \bar{X}_2 > 6.58 \mid \mu_1 - \mu_2 = 10) \\
&= P\left(\frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{4} > \frac{6.58 - 10}{4}\right) \\
&= P\left(Z > -0.855\right) \\
&= P(Z < 0.855) \approx \boxed{0.8037}.
\end{aligned}$$